$D = k\sqrt{e_{ij}e_{ij}}$.

In the case of plane deformation, all conditions of plasticity are reduced to a single condition $\sigma_1 - \sigma_2 = 2k$, $\sigma_1 > \sigma_2$. The dissipation function is then

$$D = k\sqrt{2e_{ij}e_{ij}} = k\sqrt{2}\,(e_x^2 + e_y^2 + 2e_{xy}^2)^{1/2} \tag{17}$$

From (11) and (17) we obtain

$$\sigma_x' = ke_x\sqrt{2/J_2}, \quad \sigma_y' = ke_y\sqrt{2/J_2}, \quad \tau_{xy} = ke_{xy}\sqrt{2/J_2}. \tag{18}$$

Substituting (18) into the equations of equilibrium and adding the equation expressing the incompressibility, i.e. $e_x + e_y = 0$, we obtain finally the equations which we intended to derive

Other particular cases can be investigated in an analogous manner.

The authors thank G. I. Bykovtsev for his valuable comments.

BIBLIOGRAPHY

1. Prager, V., Problems of the Theory of Plasticity. Moscow, Fizmatgiz, 1958.
2. Ivlev, D. D., Theory of Perfect Plasticity. Moscow, "Nauka", 1966.

Translated by J. M. S.

ON THE APPROXIMATE SOLUTION OF PROBLEMS OF LINEAR VISCOELASTICITY

PMM Vol. 35, №1, 1971, pp. 185-189

A. B. EFIMOV and V. I. MALYI
(Moscow)

(Received July 24, 1970)

An approximate solution of problems of linear viscoelasticity is derived. The method is applicable to both ageing and nonageing materials, as well as in numerical solution of related problems of elasticity. An estimate is made of the accuracy of the derived solution. The problem of a ponderable viscoelastic hemisphere lying on a horizontal smooth base is given as an example.

The solution of quasi-static problems of linear viscoelasticity for bodies with stationary boundaries reduces to the interpretation of the operator functions of viscoelasticity [1–3]. In the case of an isotropic material the viscoelastic properties are defined by two operators: $\mathbf{E}$ and $\boldsymbol{\nu}$. The dependence of the solution on operator $\mathbf{E}$, which can be determined by uncomplicated experiments on creep or relaxation, is simple. The dependence on operator $\boldsymbol{\nu}$ whose experimental determination is considerably more difficult is not negligible.

If the dependence of a solution on the Poisson ratio is complex, it is possible to obtain it by method of approximations [3–5].

1. Let us consider an arbitrary parameter of the stress-strain state $f(\mathbf{r}, \nu, t)$ of stressed elastic body whose dependence on time is determined by the variation of boundary conditions with time. Solution of the related problem of viscoelasticity is obtained by the substitution in the function f of operator $\boldsymbol{\nu}$ for the constant ν. The exact determination of $f(\mathbf{r}, \boldsymbol{\nu}, t)$ is cumbersome, if at all possible. Let us construct an approximate solution.

The solution of an elasticity problem is an analytic function of the complex variable ν in the band $(-1 < \mathrm{Re}\,\nu < 1/2)$ [6, 7], hence, in the vicinity of any point $\nu_0 \in (-1, 1/2)$ it can be expressed in terms of the Taylor expansion

$$f(\mathbf{r}, \nu, t) = \sum_{l=0}^{\infty} (\nu - \nu_0)^l a_l(\mathbf{r}, \nu_0, t) \tag{1.1}$$

In the general case of ageing materials the operator $\boldsymbol{\nu}$ is expressed by

$$\boldsymbol{\nu}\varphi(t) = \nu_1(t)\varphi(t) + \int_0^t K_\nu(t, \tau)\varphi(\tau)\,d\tau$$

It follows from (1.1) that the solution of the viscoelasticity problem can be reduced to the determination of arbitrary powers of the integral operator. Since for large l the difficulties arising in this process are of the same order of magnitude as in the direct determination of function $f(\mathbf{r}, \boldsymbol{\nu}, t)$, we shall limit our analysis to the first m terms of the expansion (1.1). Let us estimate the accuracy of the derived approximate solution $f_m(\mathbf{r}, \boldsymbol{\nu}, t)$

$$\max_t |f(\mathbf{r}, \boldsymbol{\nu}, t) - f_m(\mathbf{r}, \boldsymbol{\nu}, t) \leqslant \quad (0 \leqslant t \leqslant T) \leqslant$$

$$\leqslant \sum_{l=m+1}^{\infty} \max_t |(\boldsymbol{\nu} - \nu_0)^l a_l(\mathbf{r}, \nu_0, t)| \leqslant \sum_{l=m+1}^{\infty} \|\boldsymbol{\nu} - \nu_0\|^l \|a_l(\mathbf{r}, \nu_0, t)\| \tag{1.2}$$

where the following notation is used for integral operators and functions [9]:

$$\|\boldsymbol{\nu} - \nu_0\| = \max_t \left[|\nu_1(t) - \nu_0| + \int_0^t |K_\nu(t, \tau)|\,d\tau \right]$$

$$\|a_l(\mathbf{r}, \nu_0, t)\| = \max_t |a_l(\mathbf{r}, \nu_0, t)| \qquad (0 \leqslant t \leqslant T) \tag{1.3}$$

For complex arguments ν along the circumference $|\nu - \nu_0| = r$, lying entirely in the region analyticity of function f the Taylor's coefficients a_l, expressed in terms of function f, are

$$a_l(\mathbf{r}, \nu_0, t) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(\mathbf{r}, \nu, t) - f_m(\mathbf{r}, \nu, t)}{(\nu - \nu_0)^{l+1}}\,d\nu \qquad (l > m)$$

This yields the estimate for Taylor's coefficients a_l

$$|a_l(\mathbf{r}, \nu_0, t)| \leqslant \frac{\Delta(\mathbf{r}, \nu_0, t)}{r^l} \qquad (l > m)$$

$$\Delta(\mathbf{r}, \nu_0, t) = \max_\nu |f(\mathbf{r}, \nu, t) - f_m(\mathbf{r}, \nu, t)| \qquad (\nu \in \Gamma_r)$$

and then, for expansion (1.2)

$$\max_t |f(\mathbf{r}, \boldsymbol{\nu}, t) - f_m(\mathbf{r}, \boldsymbol{\nu}, t)| \leqslant M(\mathbf{r}, \nu_0) \frac{r}{r - \|\boldsymbol{\nu} - \nu_0\|} \left(\frac{\|\boldsymbol{\nu} - \nu_0\|}{r} \right)^{m+1}$$

$$M(\mathbf{r}, \nu_0) = \max_t \Delta(\mathbf{r}, \nu_0, t) \qquad (0 \leqslant t \leqslant T) \tag{1.4}$$

Thus the error of the approximate solution $f_m(\mathbf{r}, \boldsymbol{\nu}, t)$ can be readily estimated, if the solution $f(\mathbf{r}, \nu, t)$ for the elasticity problem has been calculated along the circumference Γ_r in the complex plane ν. The accuracy of the estimate increases with increasing number of terms of Taylor's expansion retained in the approximation.

If the external stresses vary in proportion to a single parameter $\psi(t)$ or are a linear combination of k one-parameter stresses with parameters $\psi_j(t)$ $(1 \leqslant j \leqslant k)$, the dependence of the solution of the viscoelasticity problem on parameter $\mathbf{v}$ is of the form

$$f(\mathbf{r}, \mathbf{v}, t) = \sum_{j=1}^{k} f_j(\mathbf{r}, \mathbf{v}) \psi_j(t)$$

and instead of analyzing function $f(\mathbf{r}, \mathbf{v}, t)$ we analyze each operator $f_j(\mathbf{r}, \mathbf{v})$. Estimate (1.4) then becomes the estimate for the norm of the residue operator $\varphi_j(\mathbf{r}, \mathbf{v})$

$$\| \varphi_j(\mathbf{r}, \mathbf{v}) \| \leqslant M_j(\mathbf{r}, \nu_0) \frac{r}{r - \| \mathbf{v} - \nu_0 \|} \left(\frac{\| \mathbf{v} - \nu_0 \|}{r} \right)^{m_j + 1} \qquad (1.5)$$

$$\varphi_j(\mathbf{r}, \mathbf{v}) = f_j(\mathbf{r}, \mathbf{v}) - \sum_{i=0}^{m_j} a_i(\mathbf{r}, \nu_0)(\mathbf{v} - \mathbf{v}_0)^i$$

$$M_j(\mathbf{r}, \nu_0) = \max_\nu | \varphi_j(\mathbf{r}, \nu) | \qquad (\nu \in \Gamma_r)$$

Similar estimates can be easily derived by expanding the solution in terms of a certain analytic function of Poisson's ratio.

2. Let us consider, as an example, the problem of a ponderous hemisphere lying on a smooth base. The material of the hemisphere is assumed to be linearly viscoelastic and ageing.

A simple substitution in the equations and boundary conditions will show that the expression

$$u_\alpha(\mathbf{r}, t) = \gamma R^2 \mathbf{E}^{-1} u_\alpha^\circ(\mathbf{r}, \mathbf{v}) \, \theta(t)$$

is the solution of the problem of viscoelasticity, if $\gamma R^2 E^{-1} u_\alpha^0(\mathbf{r}, \mathbf{v})$ is the solution of the related elasticity problem. Here γ is the specific gravity of the material, R is the hemisphere radius, $\mathbf{E}$ is Young's operator, and $\theta(t)$ is the Heaviside unit function. For ageing materials the Poisson operator $\mathbf{v}$ is in this case expressed in terms of bulk and shear operators $\mathbf{K}$ and $\mathbf{G}$ as follows:

$$\mathbf{v} = \tfrac{1}{2}(3\mathbf{K} - 2\mathbf{G})(3\mathbf{K} + \mathbf{G})^{-1}$$

Let us estimate the dimensionless displacement $u_\alpha^0(\mathbf{r}, \mathbf{v})\,\theta(t)$. According to (1.5) we have

$$\max_t | u_\alpha^\circ(\mathbf{r}, \mathbf{v})\,\theta(t) - u_\alpha^\circ(\mathbf{r}, \mathbf{v}_0) | \leqslant \| u_\alpha^\circ(\mathbf{r}, \mathbf{v}) - u_\alpha^\circ(\mathbf{r}, \nu_0) \| \leqslant \qquad (0 \leqslant t \leqslant T)$$

$$\leqslant M_\alpha(\mathbf{r}) \frac{\| \mathbf{v} - \nu_0 \|}{r - \| \mathbf{v} - \nu_0 \|} = \delta(\mathbf{r}, T) \qquad M_\alpha(\mathbf{r}) = \max_\nu | u_\alpha^\circ(\mathbf{r}, \nu) - u_\alpha^\circ(\mathbf{r}, \nu_0) | \quad (\nu \in \Gamma_r$$

Thus for estimating the accuracy of the approximate solution $u_\alpha^\circ(r, \nu_0)$ it is sufficient to construct the solution of the elasticity problem $u_\alpha^0(\mathbf{r}, \mathbf{v})$ for ν along the complex circumference Γ_r

$$\nu = \nu_0 + r(\cos\varphi + i \sin\varphi)$$

which is not difficult when the solution of the elasticity problem is given by an analytic expression and, also, when it is to be determined numerically.

Let us, for example, estimate the displacement of the hemisphere top A. For $\nu_0 = 0.25$ we have [8]

$$u^\circ(A, \nu_0) = -0.426$$

Values of $\operatorname{Re} u^\circ(A, \nu)$ and $\operatorname{Im} u^0(A, \nu)$ calculated by the same numerical method for certain values of φ (in degrees) along the circumference Γ_r $(r = 0.25)$ are tabulated below

$\varphi = 0$	18	36	54	72	90
$-\operatorname{Re} u^\circ = 0.439$	0.439	0.437	0.435	0.431	0.427
$-\operatorname{Im} u^\circ = 0$	0.0037	0.0073	0.0104	0.0126	0.0135

$\varphi = 108$	126	144	162	180
$-\operatorname{Re} u^\circ = 0.423$	0.419	0.416	0.414	0.413
$-\operatorname{Im} u^\circ = 0.0131$	0.0112	0.0082	0.0043	0

In this case the parameter $M(A)$ is equal 0.014. The deviation of the dimensionless values of the viscoelastic $u^\circ(A, \nu)\,\theta(t)$ from the elastic displacement $u^\circ(A, \nu_0)$ is estimated as follows:

$$\max_t |u^\circ(A, \nu)\,\theta(t) - u^\circ(A, \nu_0)| \leqslant \frac{M(A)\,\|\nu - \nu_0\|}{r - \|\nu - \nu_0\|} \leqslant$$

$$\leqslant \frac{0.014\,\|\nu - \nu_0\|}{0.25 - \|\nu - \nu_0\|} = \delta(A, \|\nu - \nu_0\|) \qquad (0 \leqslant t \leqslant T) \tag{2.1}$$

This estimate makes it possible to evaluate the dimensionless displacement as a function of time without knowning the specific form of operator ν.

Values of δ for several $\Delta = \|\nu - \nu_0\|$ are given below.

$\Delta = 0.05$	0.10	0.15	0.20
$\delta = 0.0035$	0.0093	0.021	0.056

Let us estimate the actual displacements

$$u_\alpha(\mathbf{r}, T) = \gamma R^2 \mathrm{E}^{-1} u_\alpha^\circ(\mathbf{r}, \nu)\,\theta(t)$$

Using the integral form of the operator

$$\mathrm{E}^{-1}\varphi(t) = \frac{\varphi(T)}{E_0(T)} + \int_0^T K(T, t)\,\varphi(t)\,dt$$

and estimate (2.1), we obtain

$$|u_\alpha(\mathbf{r}, T) - \gamma R^2 \mathrm{E}^{-1} u_\alpha^\circ(\mathbf{r}, \nu_0)\,\theta(T)| = \gamma R^2\,|\mathrm{E}^{-1}[u_\alpha^\circ(\mathbf{r}, \nu) - u_\alpha^\circ(\mathbf{r}, \nu_0)]\,\theta(T)| \leqslant \tag{2.2}$$

$$\leqslant \gamma R^2 \eta(T)\,\delta(\mathbf{r}, T)$$

$$\eta(T) = \frac{1}{E_0(T)} + \int_0^T |K(T, t)|\,dt$$

From the inequality in (2.2) directly follows the estimate of displacements

$$\varepsilon(T)\,u_\alpha^\circ(\mathbf{r}, \nu_0) - \eta(T)\,\delta(\mathbf{r}, T) \leqslant \frac{u_\alpha(\mathbf{r}, T)}{\gamma R^2} \leqslant \varepsilon(T)\,u_\alpha^\circ(\mathbf{r}, \nu_0) + \eta(T)\,\delta(\mathbf{r}, T)$$

where $\varepsilon(T)$ is a function of creep. If $K(T, t) \geqslant 0$, which is true for at least nonageing materials, then $\eta(T)$ coincides with the function of creep $\varepsilon(T)$, and the estimate of displacements derived in the approximate solution

$$\gamma R^2 \mathrm{E}^{-1} u_\alpha^\circ(\mathbf{r}, \nu_0)\,\theta(T)$$

reduces to the simpler form

$$\varepsilon(T)\,[u_\alpha^\circ(\mathbf{r}, \nu_0) - \delta(\mathbf{r}, T)] \leqslant \frac{u_\alpha(\mathbf{r}, T)}{\gamma R^2} \leqslant \varepsilon(T)\,[u_\alpha^\circ(\mathbf{r}, \nu_0) + \delta(\mathbf{r}, T)]$$

The authors thank L. S. Barkov and G. N. Aleksandrov for their help in numerical calculations.

BIBLIOGRAPHY

1. Volterra, V., Leçons sur les fonctions de lignes. Paris, 1913.
2. Rabotnov, Iu. N., Equilibrium of an elastic medium with aftereffect. PMM Vol. 12, №1, 1948.

3. Il'iushin, A. A. and Pobedria, B. E., Fundamentals of the Theory of Thermo-viscoelasticity. "Nauka", Moscow, 1969.
4. Il'iushin, A. A., The method of approximation for the calculation (stressing) of structures by the linear theory of thermo-viscoelasticity. Mekhanika Polimerov, №2, 1968.
5. Bykov, D. L., On one method of determination of stresses and strains in linearly viscoelastic bodies. Inzh. Zh. MTT, №2, 1968.
6. Mikhlin, S. G., Further investigation of Cosserat's functions. Vestn. LGU, ser. Maths, Mech. and Astronomy Vol. 7, №2, 1967.
7. Pobedria, B. E., On the solvability of problems of the theory of elasticity of contact type. PMM Vol. 33, №4, 1969.
8. Bondareva, V. F., The contact problem of a ponderous hemisphere. Theoretical Investigations in the Domain of Physics and Metrology. Transactions of the Metrological Inst. of SSSR, №119(199), Izd. standartov (Russian standards), Moscow, 1970.
9. Kantorovich, L. V. and Akilov, G. P., Functional Analysis in Normalized Spaces. Fizmatgiz, Moscow, 1959.

Translated by J. J. D.

ON SELF-SIMILAR SOLUTIONS OF THE SECOND KIND IN THE THEORY OF UNSTEADY FILTRATION

PMM Vol. 35, №1, 1971, pp. 189-192
V. I. KERCHMAN
(Moscow)
(Received February 16, 1970)

If a self-similar solution is to be an asymptotic representation of a specific class of not self-similar motions, it must be stable with respect to small perturbations. Proof of the stability of self-similar solutions of the second kind of the Cauchy problem is given in linear approximation for the equation of elastic-plastic filtration mode derived in [1]. The solution of a similar axisymmetric problem is constructed.

1. As shown in [1], the self-similar solution of the Cauchy problem for one-dimensional equation of elastic-plastic filtration

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial u}{\partial t}\right)\frac{\partial^2 u}{\partial x^2}, \qquad a^2(z) = a_1^2\ (z < 0); \qquad a_2^2\ (z > 0) \tag{1.1}$$

is of the form

$$U_0 - u(x, t) = \frac{A}{(a_1^2 t)^{1/2(1+\alpha)}} f(\xi), \qquad \xi = \frac{x}{\sqrt{a_1^2 t}} \tag{1.2}$$

Here function f is expressed in terms of parabolic cylinder functions determined by the system of equations

$$D_{\alpha+2}(\xi_0 / \sqrt{2}) = 0, \quad M(-1 - {}^1/_2\alpha, {}^1/_2; {}^1/_4\xi_0^2\varepsilon^{-1}) = 0, \quad \varepsilon = a_2^2 / a_1^2 \tag{1.3}$$

with the exponent α and the value of $\xi = \xi_0$ such that $\partial u / \partial t = 0$ when $x = x_0(t) = \xi_0 \sqrt{a_1^2 t}$.

Let us consider the solution of a Cauchy problem with initial data defined at a certain instant t_0 are defined by the weakly perturbed self-similar solution